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(Thanks to Khoa Nguyen) This is a neat way to prove that if $\alpha \in \mathcal{D}(\mathbb{R})$, is a distribution, with distributional derivative $\alpha' = 0$, then $\alpha = c$ (all in the distributional sense). For any $\varphi \in C_0^\infty(\mathbb{R})$, our assumption implies that

$$\alpha\left(\frac{\partial \varphi}{\partial x}\right) = 0$$

Let $\psi \in C_0^\infty(\mathbb{R})$ with $\int \psi = 0$. Then we have that because the total integral is zero, for large enough R

$$\int_{-R}^{x} \psi(s)ds \in C_0^\infty(\mathbb{R})$$

$$\frac{\partial}{\partial x}\int_{-R}^{x} \psi(s)ds = \psi(x)$$

Thus, we have shown that if a compactly supported function has integral zero, then it is the derivative of another compactly supported function. Now, choosing any $\chi \in C_0^\infty(\mathbb{R})$ with $\int \chi = 1$ we have that

$$\alpha(\psi) = \alpha\left(\psi - \left(\int_{\mathbb{R}} \psi\right)\chi + \left(\int_{\mathbb{R}} \psi\right)\chi\right)$$

Now, noting that

$$\int_{\mathbb{R}}\left(\psi - \left(\int_{\mathbb{R}} \psi\right)\chi\right) = 0$$

by choice of χ, we have that

$$\psi - \left(\int_{\mathbb{R}} \psi\right)\chi = \frac{\partial \eta}{\partial x}$$

for $\eta \in C_0^\infty(\mathbb{R})$. Thus, we have shown that

$$\alpha(\psi) = \alpha\left(\frac{\partial \eta}{\partial x}\right) + \int \phi \cdot \alpha(\chi) = \int_{\mathbb{R}} \alpha(\chi)\psi$$

Thus, by definition of distributions, we see that $\alpha = \alpha(\chi) = c$

This idea can be continued to higher derivatives, for example $\alpha'' = 0$ implies that $\alpha = ax + b$ (as a distribution).